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Semi-analytical model of librations of a rigid moon orbiting an oblate planet

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Abstract. In this paper we study the rotational motion of a rigid satellite moving in the gravitational field of an oblate planet. The librations of the moon around the equilibrium state of synchronous stable rotation are analyzed by the application of the normal forms theory. We construct a general, semi-analytical theory of librations and we show its application to a few small moons of Saturn and Jupiter.

Key words: celestial mechanics, stellar dynamics – planets and satellites: general

1. Introduction

It is well known that many natural satellites of big planets evolved tidally to the state of synchronous rotation (Wisdom, 1987). Among them, we can distinguish a group of equatorial satellites. To this group belong Jupiter's moons: Metis, Adrastea, Amalthea; Galilean moons Io, Europa, Ganymede and Callisto; Saturn's satellites: Pan, Atlas, Prometheus, Pandora, Epimetheus, Janus, Enceladus, Dione, Rhea; Uranus satellites: Ariel, Umbriel, Titania and Oberon. Orbits of all these satellites are almost perfectly circular—they have eccentricities smaller that 0.01 and inclinations smaller than 0.5 deg. In this paper we consider some of the moons whose shapes are highly irregular. The inertial and orbital data for them are given in Table 2.

The aim of this paper is to build a model of rotational motion which is applicable to such a class of small orbiters of the giant planets. In this model the parent planet is represented as a spheroid, i.e., a flattened body possessing an axis of symmetry. For the selected moons this assumption seems to be natural and obvious. Because they move relatively close to the mother planet, its oblateness may affect substantially their rotation. Most previous studies of the rotational motion of a rigid satellite assumed that the central body is a point mass or a sphere of uniform density. In this formulation the problem was investigated by many authors (see, e.g., Kondurar, 1959; Beletskii, 1965, 1975; Kinoshita, 1972b,a; Barkin, 1972; Markeev, 1978, 1985; Chermnych, 1987; Kokoriev & Kirpichnikov, 1988; Celetti, 1990a,b; Wang et al., 1992; Shevtshenko & Sokolsky, 1995; Sushko & Shevtshenko, 1996). The model presented here may be considered as a natural generalization of the classical approach.

In this paper the model dynamics is described in the frames of Hamiltonian formalism. The synchronous rotation of a moon is an equilibrium of Hamiltonian equations of motion, thus the motions in its vicinity (librations) can be investigated effectively by the application of the normal forms theory. The semi-analytical, local theory of librations derived with this approach was tested on a few small, very irregular moons of Jupiter and Saturn. We calculated their inertial characteristics using discrete topography models by Stooke & Lumsdon (1993); Stooke (1993a,b, 1994).

2. The Hamiltonian function

Let us assume that the moon's mass center moves in the equatorial plane of a spheroidal planet, i.e., a flattened, axially symmetric body. The planet rotates uniformly around the symmetry axis (see Fig. 1). We assume also that the orbit of the moon is circular. According to Barkin (1985), it is a restricted problem of rotational motion because the relative, orbital motion of the bodies is fixed. It is convenient to describe the rotational motion of the moon with respect to the orbital frame. The orbital reference frame $\mathcal{O}$ is defined by versors $\left\{\widehat{\boldsymbol{r}},\widehat{\boldsymbol{t}},\widehat{\boldsymbol{n}}\right\}$ which have directions of the radius vector of the moon, the orbital velocity and the normal to the orbital plane, respectively. The body fixed, principal axes frame $\mathcal{B}$ is determined by versors $\{\widehat{\boldsymbol{u}},\widehat{\boldsymbol{v}},\widehat{\boldsymbol{w}}\}$.

The orientation of the body frame $\mathcal{B}$ with respect to the orbital frame $\mathcal{O}$ is given by the attitude matrix $\mathbf{A}$. It is defined in the following way. If, for a vector $\boldsymbol{x}$, its coordinates in the orbital and the body frame we denote by $\mathbf{x}=[x_1,x_2,x_3]^T$, and $\mathbf{X}=[X_1,X_2,X_3]^T$, respectively, then the attitude matrix $\mathbf{A}$ transforms coordinates $\mathbf{x}$ to $\mathbf{X}$ according to the rule

$$\mathbf{X}=\mathbf{A}\mathbf{x}. \tag{1}$$

Thus, $\mathbf{A}=[\widehat{\mathbf{u}},\widehat{\mathbf{v}},\widehat{\mathbf{w}}]^T$, and at the same time $\mathbf{A}=[\widehat{\mathbf{R}},\widehat{\mathbf{T}},\widehat{\mathbf{N}}]$, i.e., columns of the attitude matrix are equal to the coordinates of orbital reference frame versors.

For parameterization of the attitude matrix $\mathbf{A}$ we use the Euler angles of the type 3-2-1 (see Markley, 1978). In terms of

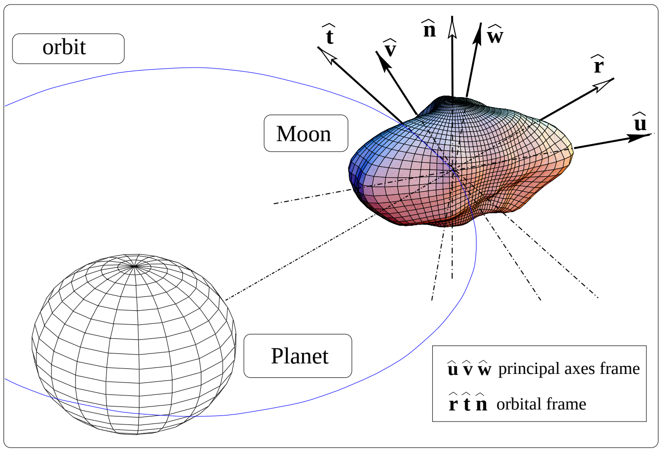

Fig. 1. Geometry of the model.

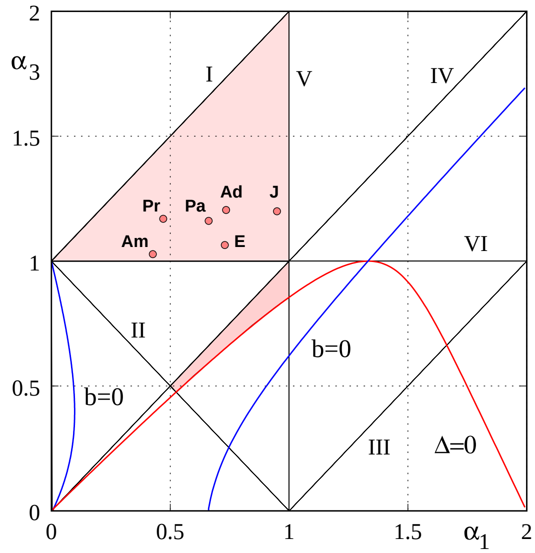

Fig. 2. Diagram of the linear stability of synchronous rotation for the fixed value $\epsilon = 0.0015$. Dots represent values of α_1, α_3 derived from the estimation of the main moments of inertia made by the authors (see Sect. 4.1). Here we denoted: Am – Amalthea, Pr – Prometheus, Pa – Pandora, Ad – Adrastea, E – Epimetheus, J – Janus.

these angles we have

$$\widehat{\mathbf{R}} = \begin{bmatrix} \cos\phi\cos\theta \\ -\cos\psi\sin\phi + \cos\phi\sin\psi\sin\theta \\ \sin\phi\sin\psi + \cos\phi\cos\psi\sin\theta \end{bmatrix}, \tag{2}$$

and

$$\widehat{\mathbf{N}} = \begin{bmatrix} -\sin\theta \\ \cos\theta\sin\psi \\ \cos\psi\cos\theta \end{bmatrix}. \tag{3}$$

Calculation of the potential of gravitational interaction of a spheroid and a rigid body can be done in a few different ways (Maciejewski, 1995, 1997).

The gravitational potential of a spheroidal planet has the form

$$V_{\mathrm{P}} = -\frac{\mu}{r}\left\{1 - \frac{1}{2}J_2\left(\frac{a_{\mathrm{E}}}{r}\right)^2\left[3\left(\frac{z}{r}\right)^2 - 1\right] + \ldots\right\}, \tag{4}$$

where $\mu = GM_{\mathrm{P}}$ is the mass parameter of the planet, a_{E} is its equatorial radius, J_2 represents the second harmonic, r is the orbital radius (Aksenov, 1977). To calculate the potential of interaction between the planet and the moon we have to integrate the potential V_{P} over the moon's body. This integral can be computed effectively if we expand the integrated function into series, assuming that the dimensions of the moon are much smaller than the orbital radius. In this way we obtain the potential of the planet-moon interaction also in the form of series. The first terms of the series affecting the rotational dynamics are given by the formula

$$V = \frac{3}{2}\frac{\mu}{r^3}(1+5\epsilon)\langle\widehat{\mathbf{R}}, \mathbf{I}\widehat{\mathbf{R}}\rangle - 3\epsilon\frac{\mu}{r^3}\langle\widehat{\mathbf{N}}, \mathbf{I}\widehat{\mathbf{N}}\rangle, \tag{5}$$

where $\mathbf{I} = \mathrm{diag}\,[A, B, C]$, and A, B, C denote principal moments of inertia of the moon; by $\langle\mathbf{a}, \mathbf{b}\rangle$ we denote the scalar product of vectors $\mathbf{a}$ and $\mathbf{b}$. The terms proportional to the parameter

$$\epsilon = \frac{1}{2}J_2\left(\frac{a_{\mathrm{E}}}{r}\right)^2, \tag{6}$$

describe the influence of the planet oblateness on the rotation of the moon. The mean orbital motion of the moon in a circular orbit of a radius r is

$$\omega = \omega_{\mathrm{K}}\sqrt{1+3\epsilon}, \quad \text{where} \quad \omega_{\mathrm{K}}^2 = \frac{\mu}{r^3}. \tag{7}$$

The frequency ω_{K} denotes Keplerian mean motion of the moon in the field of a point mass center. Let us remark here that in (Maciejewski, 1997) the above expression for the potential was obtained in a simpler way thanks to the application of the generalized two centers problem.

The Hamiltonian of the problem can be obtained with the help of standard techniques. In order to limit the number of parameters we chose the unit of time in such a way that $\omega_{\mathrm{K}} = 1$. We also re-scale the generalized momenta by B to eliminate one additional parameter. Finally we obtain a time independent Hamiltonian of the form

$$H(\mathbf{q}, \mathbf{p}) = T(\mathbf{q}, \mathbf{p}) + V(\mathbf{q}), \tag{8}$$

where $\mathbf{q} = [q_1, q_2, q_3]^T = [\phi, \theta, \psi]^T$ are the generalized coordinates, and $\mathbf{p}$ are the conjugate momenta. The kinetic energy T, following Maciejewski (1987), is given by

$$\begin{aligned} T \;=\; & \frac{1}{2}\left[\frac{(p_1 + p_3\sin q_2)\sin q_3 + p_2\cos q_2\cos q_3}{\cos q_2}\right]^2 \\ & + \frac{1}{2\alpha_3}\left[\frac{(p_1 + p_3\sin q_2)\cos q_3 - p_2\cos q_2\sin q_3}{\cos q_2}\right]^2 \\ & + \frac{1}{\alpha_1}p_3^2 - \omega p_1, \end{aligned} \tag{9}$$

where $\alpha_1 = A/B$, $\alpha_3 = C/B$. After rescaling the parameters the potential $V(\mathbf{q})$ has the following form

$$V(\mathbf{q}) = \frac{3}{2}(1+5\epsilon)\langle\widehat{\mathbf{R}}, \mathbf{I}\widehat{\mathbf{R}}\rangle - 3\epsilon\langle\widehat{\mathbf{N}}, \mathbf{I}\widehat{\mathbf{N}}\rangle, \tag{10}$$

where now $\mathbf{I} = \mathrm{diag}[\alpha_1, 1, \alpha_3]$. From general considerations of the problem of two rigid bodies one may state that the Hamiltonian system (8) possesses a class of equilibria solutions, the so called Lagrange equilibria (Barkin, 1985). They correspond to fixed orientations of the moon in the orbital reference frame, at which the directions of principal axes of inertia coincide with the directions of the orbital frame. We are interested in one of these solutions, which is specified by the following values of phase variables

$$\mathbf{x}^0 = \left\{ q_1^0 = q_2^0 = q_3^0 = p_2^0 = p_3^0 = 0, \quad p_1^0 = \alpha_3\sqrt{1+3\epsilon} \right\}. \tag{11}$$

The stability of this solution, which we shall call the synchronous rotation state, is examined in the next section.

Let us note that the case of a spherical planet (i.e., the case $\epsilon = 0$) was investigated analytically by Sushko & Shevtshenko (1996) and in some details in (Goździewski & Maciejewski, 1990; Maciejewski & Goździewski, 1992, 1995)

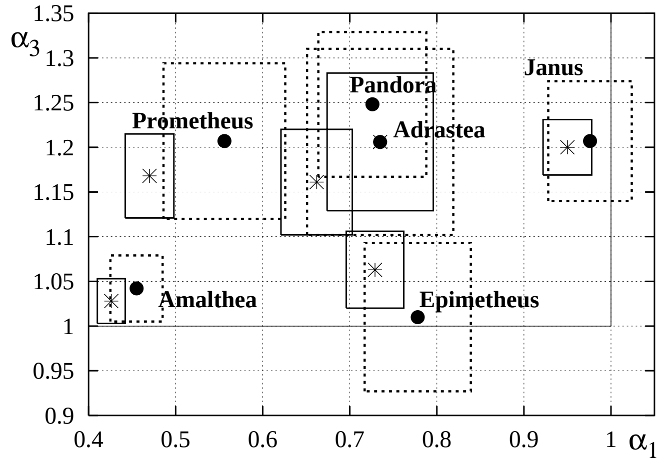

Fig. 3. Comparison of the ratios α_1 and α_3, calculated from the harmonic topography expansions (asterisks) and from the best fitting ellipsoid estimations (dots). The rectangles represent the mean errors.

3. The local theory

For a systematic study of the dynamics in the neighborhood of the synchronous rotation state we apply the theory of normal forms of Hamiltonian systems around an equilibrium (Bruno, 1988). It can be considered here as a special case of Lie-Hori-Deprit perturbation theory (see, e.g., Hori, 1966; Deprit, 1969; Mersman, 1971; Cary, 1981). The whole procedure is naturally divided into two steps of linear and non-linear normalization. In the first step we simplify the linearized system. For this purpose we have to determine the stability character of the equilibrium. If the equilibrium is stable, we can be sure that a solution with the initial condition near the equilibrium will remain close to it forever. It is also well known (Arnold, 1978) that even if the equilibrium is stable linearly, almost all solutions with initial conditions near the equilibrium will spend an exponentially long time in its vicinity. These facts justify the applicability of the local theories of motion.

3.1. Analyzis of the linearized model

The qualitative character of the system dynamics in a neighborhood of an equilibrium depends on the linear part of the equations of motion. To perform the linearization we shift the origin to the equilibrium $\mathbf{x} = \mathbf{x}^0 + \mathbf{y}$. The linearized equations have the form

$$\dot{\mathbf{y}} = \mathbf{J}\mathbf{H}_2\mathbf{y}, \quad \mathbf{H}_2 = \nabla^2 H(\mathbf{x}^0), \quad \mathbf{J} = \begin{bmatrix} \mathbf{0} & \mathbf{E} \\ -\mathbf{E} & \mathbf{0} \end{bmatrix}. \tag{12}$$

The characteristic frequencies of librations $\omega_1, \omega_2, \omega_3$ are defined by the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of the matrix $\mathbf{F} = \mathbf{J}\mathbf{H}_2$. Its non-zero elements are the following

$$\begin{gathered} F_{14} = \alpha_3^{-1}, \quad F_{23} = -F_{65} = (\alpha_3 - 1)\beta, \quad F_{25} = 1, \\ F_{32} = \beta, \quad F_{36} = \alpha_1^{-1}, \quad F_{41} = (\alpha_1 - 1)(3 + 10\epsilon), \\ F_{52} = -\alpha_1\alpha_3(3 + 16\epsilon)(4 + 19\epsilon), \quad F_{56} = -\beta, \\ F_{63} = (1 - \alpha_3)[\alpha_3 + 3(2 + \alpha_3)\epsilon)], \end{gathered}$$

where $\beta = \sqrt{1+3\epsilon}$. The characteristic polynomial of the matrix $\mathbf{F}$ has the form

$$D_\lambda = \det(\mathbf{I}\mathbf{H}_2 - \lambda\mathbf{E}) = (\lambda^2 + a)(\lambda_4 + b\lambda_2 + c) = 0, \tag{13}$$

where

$$\begin{aligned} c &= \frac{(-1+\alpha_3)(-\alpha_1+\alpha_3)(1+9\epsilon)(4+19\epsilon)}{\alpha_1}, \\ b &= \frac{2\alpha_1 - 3\alpha_1^2 - \alpha_3 + 2\alpha_1\alpha_3 + \alpha_3^2}{\alpha_1} - \\ &\quad \epsilon\frac{16\alpha_1^2 - \alpha_1(6+13\alpha_3) - 3(-2+\alpha_3+\alpha_3^2)}{\alpha_1}, \\ a &= \frac{(1-\alpha_1)(3+10\epsilon)}{\alpha_3}. \end{aligned}$$

The equilibrium is linearly stable if the eigenvalues of the matrix $\mathbf{J}\mathbf{H}_2$ are purely imaginary: $\lambda_i = \pm \mathrm{i}\omega_i, \quad i = 1, 2, 3$. Thus, the following conditions should be satisfied

$$a > 0 \quad \wedge \quad b > 0 \quad \wedge \quad c > 0 \quad \wedge \quad \Delta = b^2 - 4c \geq 0. \tag{14}$$

If it is so then the linear, characteristic frequencies of the libration may be calculated easily. The simplest form has the longitudal frequency

$$\omega_1 \equiv \omega_\phi = \sqrt{a}. \tag{15}$$

If ϵ is small (as for the analyzed moons, see Table 2), the frequency can be expanded with respect to this parameter

$$\omega_\phi = \omega_\phi^0 + \frac{5}{3}\omega_\phi^0\epsilon + O(\epsilon^2), \tag{16}$$

where ω_ϕ^0 corresponds to $\epsilon = 0$. Thus the time Δt, after which the influence of the oblateness of the parent planet causes a

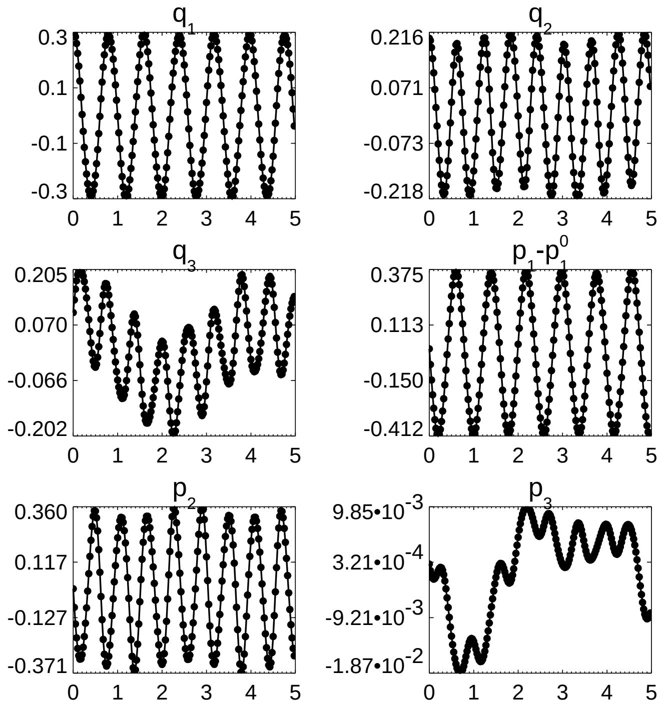

Fig. 4. Comparison of semi–analytical (dots) and numerical solutions for Amalthea over the time span of 5 orbital periods. Initial conditions: $q_1 = 0.3, q_2 = 0.2, q_3 = 0.1, p_2 = p_3 = 0, p_1 = p_1^0$.

difference in the orientation of the order 2π, may be estimated as

$$\Delta t = \frac{3}{5}\frac{1}{\omega_\phi^0}\frac{1}{\epsilon} \text{ [orbital periods]}. \tag{17}$$

Thus, as for the moons selected $\omega_\phi^0 \simeq 1$ and $\epsilon \simeq 0.001$, the time period is of the order of just a few hundred revolutions.

The linear stability analysis leads to a qualitative result similar to that obtained by Sushko & Shevtshenko (1996), who analyzed the case $\epsilon = 0$. It is illustrated in Fig. 2. The choice of $\epsilon = 0.0015$ follows from Table 2: it is almost the same for all of the moons analyzed. Fixing ϵ, we may describe the linear stability region in the plane (α_1, α_2). Their physically accepted range follows from the triangle rule, which should be satisfied by the principal moments of inertia:

$$A + B \geq C, \quad A + C \geq B, \quad B + C \geq A. \tag{18}$$

Thus the physical values of α_1 and α_3 are limited by the lines (see Fig. 2)

$$\text{I. } \alpha_3 - \alpha_1 = 1, \quad \text{II. } \alpha_1 + \alpha_3 = 1, \quad \text{III. } \alpha_1 - \alpha_3 = 1. \tag{19}$$

The border lines denoted as IV,V, VI in Fig. 2 are defined by $a = 0$ and $c = 0$. The curves of $b = 0$ and $\Delta = 0$ were determined numerically for the fixed value of ϵ. It is clear now that the inertial parameters of the moons presented in Table 2 locate them in the triangular area of the linear stability, which does not change with variations of ϵ. Especially interesting cases are

Table 1. Estimation of the orbital period T_o and periods of librations (in days) for the linearized model.

Moon	T_o	T_ϕ	T_θ	T_ψ
Adrastea (J14)	0.29	0.36	0.61	0.19
Amalthea (JV)	0.50	0.38	2.08	0.30
Prometheus (S17)	0.61	0.52	1.06	0.35
Pandora(S15)	0.63	0.67	1.39	0.40
Janus(S1)	0.69	1.96	1.95	0.53
Epimetheus(S3)	0.69	0.79	2.87	0.49

Amalthea and Janus: they are located almost on the border of the linear stability region. It is well known (Veverka et al., 1989) that Amalthea has a very irregular shape, which cannot be described reasonably by any triaxial ellipsoid—in spite of this the moon is almost symmetric dynamically. The periods of librations in the linear approximation are given in Table 1.

3.2. Normalization procedure

Applying the normal form theory we look for such a change of variables which transforms the original Hamiltonian H into a new, much simpler Hamiltonian H^*. Equations of motion with this Hamiltonian can be solved explicitly (of course we have to make some additional assumptions). The construction of a local theory of motion in the vicinity of the equilibrium can be performed in the form of the following algorithm:

1. Take the original Hamiltonian and determine phase variables corresponding to the equilibrium.
2. Expand the Hamiltonian into Taylor series in a neighborhood of the equilibrium. This expansion has the form

$$H(\boldsymbol{x}) = \sum_{k=2}^{\infty} H_k(\boldsymbol{x}), \quad H_k(\boldsymbol{x}) = \sum_{|\boldsymbol{n}|=k} h_{\boldsymbol{n}} \boldsymbol{x}^{\boldsymbol{n}}, \tag{20}$$

where $\boldsymbol{x} = (x_1, \ldots, x_{2n})$ are local coordinates in the neighborhood of the equilibrium, n is the number of degrees of freedom and

$$h_{\boldsymbol{n}} \boldsymbol{x}^{\boldsymbol{n}} := h_{n_1 \ldots n_{2n}} x_1^{n_1} \cdots x_{2n}^{n_{2n}}. \tag{21}$$

3. Normalize the Hamiltonian by the Lie-Hori-Deprit method. The normalized non resonant Hamiltonian has the form

$$H^*(\mathbf{z}) = \sum_{j=1}^{\infty} H_{2j}^*(\mathbf{z}), \quad H_2^*(\mathbf{z}) = \sum_{l=1}^{n} \omega_i \rho_i, \tag{22}$$

where $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_n)$ denotes linear frequencies of the system, $\mathbf{z} = (z_1, \ldots, z_{2n})$ are the normal coordinates, $\rho_i = z_i z_{i+n}$ for $i = 1, \ldots, n$, and

$$H_{2j}^*(\mathbf{z}) = \sum_{|\boldsymbol{l}|=j} h_{l_1 \ldots l_n} \rho^{l_1} \cdots \rho_n^{l_n}. \tag{23}$$

To obtain this normal form we have to assume that the resonances are absent in the system, which means that equation

$$\langle \boldsymbol{k}, \boldsymbol{\omega} \rangle = 0, \qquad \boldsymbol{k} \in \mathbf{Z}^n, \tag{24}$$

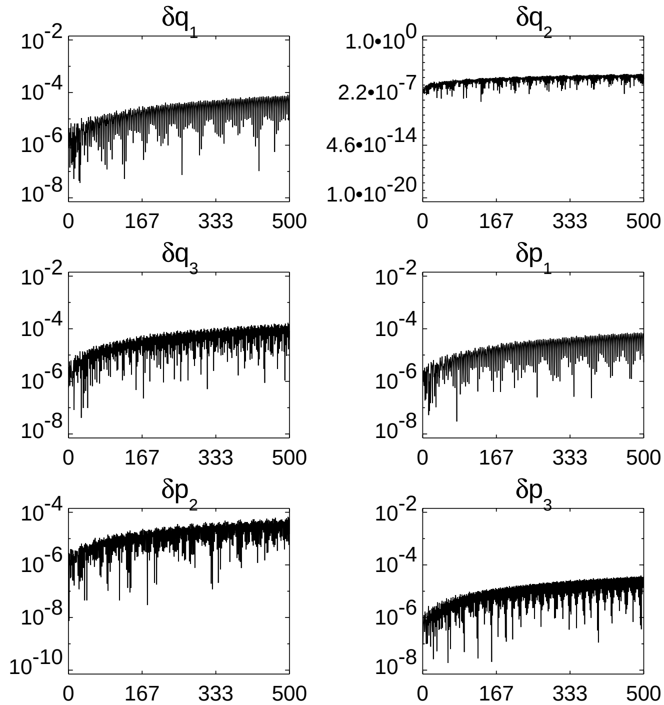

Fig. 5. Comparison of semi–analytical and numerical solutions for Epimetheus. Absolute differences between the phase variables over the time span of 500 orbital periods are shown. Initial conditions: $q_1 = q_2 = q_3 = 0.1, p_2 = p_3 = 0, p_1 = p_1^0$.

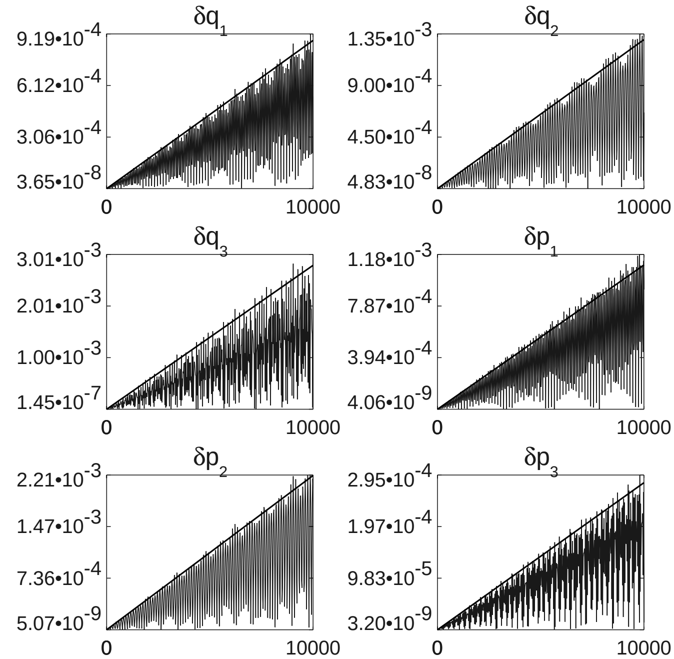

Fig. 6. Comparison of semi–analytical and numerical solutions for Amalthea. Absolute differences between the phase variables over the time span of 10000 orbital periods are shown. Initial conditions: $q_1 = q_2 = q_3 = 0.1, p_2 = p_3 = 0, p_1 = p_1^0$. The straight lines represent the result of linear fit to the points of the graphs upper bounds. The slopes at subsequent panels are approximately: 8.8×10^{-8}, 1.3×10^{-7}, 2.8×10^{-7} for coordinates and 1.1×10^{-7}, 2.2×10^{-7}, 2.8×10^{-8} for momenta.

has no non-zero solutions. In the action-angle variables $(\boldsymbol{\rho}, \boldsymbol{\phi}) = (\rho_1, \ldots, \rho_n, \varphi_1, \ldots, \varphi_n)$, where

$$\varphi_k = \frac{1}{2}(\ln z_k - \ln z_{k+n}), \quad k = 1, \ldots, n, \tag{25}$$

the normalized Hamiltonian H^* depends only on the actions: $H^* = H^*(\boldsymbol{\rho})$ (Bruno, 1988).

4. Write the solution of the averaged equations of motion. The Hamilton's equations of motion with the normalized Hamiltonian H^* have a very simple form

$$\dot{\rho}_k = -\frac{\partial H^*}{\partial \varphi_k}(\boldsymbol{\rho}) = \mathbf{0}, \quad \dot{\varphi}_k = \frac{\partial H^*}{\partial \rho_k}(\boldsymbol{\rho}), \quad k = 1, \ldots, n. \tag{26}$$

Their solution is straightforward an it is given by

$$\boldsymbol{\rho}(t) = \boldsymbol{\rho}^0 = \text{const}, \quad \boldsymbol{\phi}(t) = \boldsymbol{\Omega}(t - t_0) + \boldsymbol{\phi}^0, \tag{27}$$

where

$$\boldsymbol{\Omega} = \frac{\partial H^*}{\partial \boldsymbol{\rho}}\left(\boldsymbol{\rho}^0\right). \tag{28}$$

Generally, the frequencies $\boldsymbol{\Omega}$ are nonlinear functions of the action variables $\boldsymbol{\rho}$.

Solution (27) together with the transformation from original to normal coordinates and vice versa:

$$\boldsymbol{\rho} = \boldsymbol{\rho}(\mathbf{y}), \quad \boldsymbol{\phi} = \boldsymbol{\phi}(\mathbf{y}), \tag{29}$$

$$\mathbf{y} = \mathbf{y}(\boldsymbol{\rho}, \boldsymbol{\phi}), \tag{30}$$

give us the local theory of motion. For an initial condition $\mathbf{y}^0$ in a small neighborhood of the equilibrium using (29) we can calculate $(\boldsymbol{\rho}^0, \boldsymbol{\phi}^0)$. Then, using (27) and (30), we obtain explicit formulae for solutions of the equations of motion

$$\mathbf{y}(t) = \mathbf{y}(\boldsymbol{\rho}^0, \boldsymbol{\Omega}(t - t_0) + \boldsymbol{\phi}^0). \tag{31}$$

It must be stressed that the normalizing transformations (29) and (30) are generally divergent (Bruno, 1988), however, we can use them in practice because we study dynamics of the system over a finite interval of time.

The theory constructed above is valuable. We can use it to determine qualitative and quantitative properties of the system. For example, it can be applied, as it will be shown, to determine the dependence of the librations frequencies on the amplitudes. The analytical solution allows to compute easily the trajectories of the system, thus it is an alternative to the numerical integration procedures. It is extremely fast: the computation of the system state $\mathbf{y}(t)$ for an arbitrary time t needs just two evaluations of variables transformation and it does not depend on t. We demonstrate the properties of the analytical theory in numerical examples given below.

In our approach the resulting theory is in fact semi-analytical. Power series used in our theory have constant, numerical coefficients. In practice, for performing the normalization

Table 2. Orbital and physical parameters of small Jovian and Saturnian moons. Description: i [deg]—orbital inclination to the equatorial plane, e–eccentricity, a_{E} is the equatorial radius of the planet, r—orbital radius of the moon, A, B, C—principal moments of inertia in the units of $[mr_0^2]$ (m—mass and r_0—mean radius of the moons body). Inertial parameters $\alpha_1 = A/B$, $\alpha_3 = C/B$. Zonal harmonics: Jovian $J_2^{\mathrm{J}} \simeq 0.014736$, for Saturn, $J_2^{\mathrm{S}} \simeq 0.0165$.

Moon	i [deg]	e	r/a_E	A	B	C	α_1	α_3	ϵ
Amalthea (JV)	0.4	0.003	2.55	0.292	0.686	0.705	0.426	1.028	0.0012
Pandora (S15)	0.1	0.004	2.35	0.325	0.491	0.570	0.622	1.161	0.0015
Prometheus (S17)	0.0	0.004	2.31	0.308	0.656	0.766	0.470	1.168	0.0015
Janus (S1)	0.1/3	0.007/9	2.51	0.380	0.400	0.480	0.950	1.200	0.0013
Epimetheus (S3)	0.3/1	0.009/7	2.51	0.350	0.480	0.510	0.729	1.063	0.0013
Adrastea * (J14)	0.0	0.000	1.80	0.300	0.408	0.492	0.735	1.206	0.0023

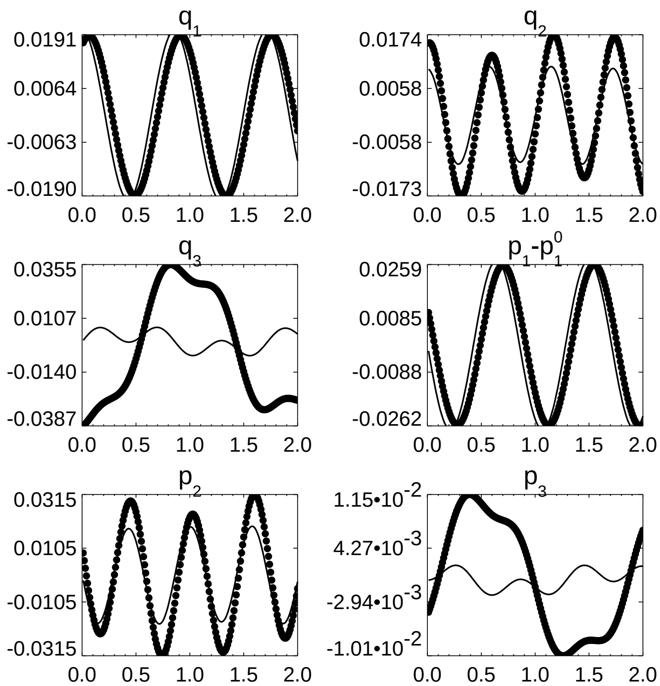

Fig. 7. Comparison of semi–analytical (dots) and numerical solutions for Prometheus over the time span of 2 orbital periods. Initial conditions: $q_1 = 0.02, q_2 = 0.01, q_3 = p_2 = p_3 = 0, p_1 = p_1^0$.

procedures, we use our software system LIE, which is especially designed for that purpose. Technical details, practical aspects, as well as examples of application of the system may be found in our papers (Goździewski & Maciejewski, 1990; Goździewski et al., 1991; Maciejewski & Goździewski, 1991, 1992, 1995).

4. Tests and discussion

4.1. Data

For tests of the theory we have chosen a number of small moons of Jupiter and Saturn (see Table 2). Their orbital elements fit very well to the assumptions of our model. To derive the inertial characteristics of the moons we used the discrete descriptions of topography constructed by Stooke & Lumsdon (1993); Stooke (1993a,b, 1994). Assuming that the density of the moons is constant, we can compute their principal moments of inertia, as well as other dynamical and geometrical characteristics. For this purpose we used the method described in (Goździewski & Maciejewski, 1995).

The data are shown in Table 2. Assuming that the mean error of topography is typically of the order of 10 kilometers, we estimated the formal error of the principal moments of inertia in the range of a few percent. It is worth noticing that the perturbation parameter ϵ is similar for all of the moons selected. According to our data it has the greatest value, among the known natural moons, for Jupiter's Metis and Adrastea. The latter we selected as an example. As we do not know the detailed topography of this moon, its inertial data were estimated on the basis of its overall dimensions, which are $12.5 \times 10 \times 7.5$ km according to Thomas (1989). We assumed that, according to Stooke & Lumsdon (1993), Prometheus and Pandora rotate synchronously. This is not consistent with other sources of data, for example (Yoder, 1995).

Under the assumption of uniform density the main moments of inertia may be calculated independently using the estimations of the moons best fitting ellipsoids. The data were published by Thomas (1989), they are also given in Slyuta & Voropayev (1997). For calculations we used the ratios of the ellipsoid axes and their mean errors. The comparison with our data is given in Fig. 3. The coincidence of parameters is far from perfect. The detailed knowledge of the topography is essential for minimizing the uncertaintity of the data. This is especially important in the cases of Epimetheus, Janus and Amalthea: the errors estimation leads to the conclusion that all the moons are located on the border of linear stability of the synchronous rotation.

4.2. Numerical tests

For the moons presented in Table 2, we constructed the theory of their librations using the algorithm described in Sect. 3. The Hamiltonian (8) was expanded in the equilibrium point with the help of MATHEMATICA up to the sixth degree. The linear and nonlinear normalizations were computed with the help of the system LIE. The obtained quasi-analytical theory was examined through direct numerical integration. First, we compared the analytical and numerical solution in the time span of 5 orbital periods for Amalthea, see Fig. 4. In spite of a relatively large initial amplitudes of the librations, the solutions agree perfectly. In the case of Epimetheus the differences be-

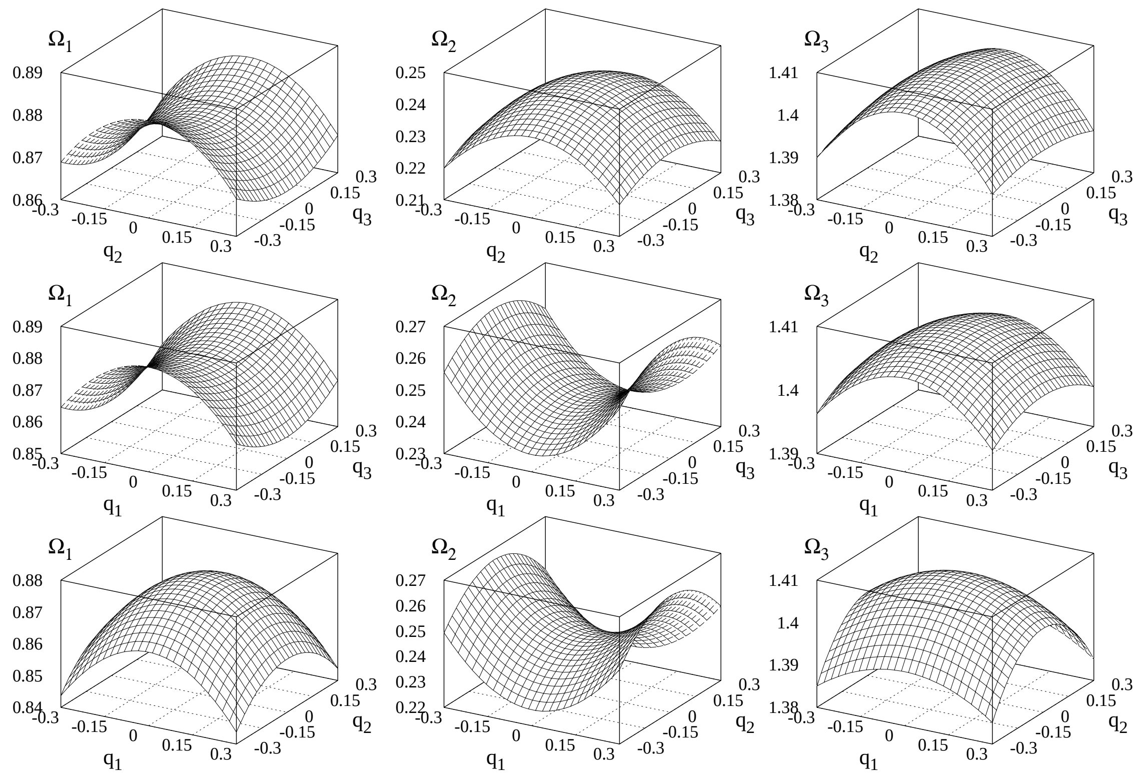

Fig. 8. Frequency–amplitude characteristics computed for Epimetheus. Initial conditions are chosen in such a way, that the values of variables, completing the coordinates in subsequent panels, are chosen as in the synchronous rotation. For example, in the first panel $q_1 = p_2 = p_3 = 0, p_1 = p_1^0$, and so on.

tween the solution derived by numerical integration and the semi-analytical result are very small in the time span of 500 orbital periods, see Fig. 5. In this test the initial amplitude of librations was $\simeq 5^0$. For a very long time duration the errors of the quasi-analytical theory grow linearly. This effect was analyzed by Sushko & Shevtshenko (1996) for the case of a spherical planet. They showed that the linear growth of errors is caused mainly by the errors of approximation of nonlinear frequencies of the librations. The same justification may by applied to our model. If we assume that the initial expansion has order $2N$, then the normalized Hamiltonian contains terms of the order N (with respect to normal variables $\boldsymbol{\rho}$) and it is approximated with the accuracy $O(|\boldsymbol{\rho}|^{(2N+1)/2})$. Then, from (28), it follows that the frequencies are determined with the errors of the order $(2N-1)/2$. In our case the nonlinear frequencies are approximated with an error $O(|\boldsymbol{\rho}|^{5/2})$. Of course, we have to assume here that the initial conditions are close enough to the equilibrium, providing the error of variables transformation of the order $O(|\boldsymbol{\rho}|^{2N+1})$. Practically we may check if the initial conditions are valid in this sense by controlling the difference after a forward and backward variables transformation.

We performed a check of the theoretical predictions of a long term behavior of the semi-analytical solution for Amalthea. This is illustrated in Fig. 6. As one can expect, in the time span of 10^4 orbital periods the difference between analytical and numerical solutions grows almost strictly linearly with time.

We build the semi-analytical theories under the assumption that there are no resonances in the system (at least, up to the order $2N$). In practice, we have to exclude the cases when the system is close to a resonance. In such events the semi-analytical theory is completely non-applicable. Surprisingly, we discovered such almost resonance cases for two of the moons: Prometheus and Janus. Their inertial and orbital parameters lead to strong resonances between the linear frequencies of the librations. For Prometheus they are

$$\begin{aligned}\sigma_{-1,-1,1} &= |-(\omega_1+\omega_2)+\omega_3| \simeq 0.008,\\ \sigma_{0,-3,1} &= |-3\omega_2+\omega_3| \simeq 0.004.\end{aligned}$$

Two of the frequencies of Janus are also almost commensurable, i.e.

$$\sigma_{-1,1,0} = |-\omega_1+\omega_2| \simeq 0.002.$$

Fig. 7 shows that the quasi-analytical non-resonant theory is not adequate for these moons. Even for a very small initial amplitude of libration, the transformation between original and normal variables fails, giving a completely wrong result. It has dramatic consequences —the semi-analytical solution does not describe the motion even qualitatively in the time span of 2 orbital periods. Investigations of the resonance effects need the construction of a different kind of local theory and are out of the scope of this paper. We plan to devote another work to them.

Finally, we investigated the dependence of nonlinear frequencies of librations on the initial amplitudes of coordinates and momenta. With the help of our theory it can be performed easily. The key is formula (28). As an example we computed frequency-amplitudes diagrams for Epimetheus (Fig. 8). The valid range of the phase variables was determined by the condition that the absolute difference between the values of original coordinates and momenta after a forward and backward transformation did not exceeded 10^{-4}. Generally, the frequencies of librations can change by a few percent when we change their amplitudes in the prescribed vicinity of the equilibrium.

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